Pergamon

Topology, Vol. 33, No. 1, pp. 181–196, 1994

0040-9383/94 $6.00 + 0.00

GEOMETRY "À LA GROMOV" FOR THE FUNDAMENTAL GROUP OF A CLOSED 3-MANIFOLD M^3 AND THE SIMPLE CONNECTIVITY AT INFINITY OF $\tilde{M}^3$

V. Poénaru

(*Received* 14 *November* 1990; *in revised form* 14 *January* 1993)

1. INTRODUCTION

In this paper, we investigate the question of the simple connectivity at infinity of the universal covering space $\tilde{M}^3$ of a closed 3-manifold M^3 with infinite, $\pi_1 M^3$. An idea which has been used already by several authors [16], [12], [23], [1] is to impose fairly general geometric type of conditions to $\pi_1 M^3$ and deduce from them that $\pi_1^\infty \tilde{M}^3 = 0$. When we talk about "geometry" here, we refer to the length-structure of the Cayley graph of $\pi_1 M^3$, with respect to a given system of generators [9]. Typically, the conditions imposed in the various papers cited above always include the hyperbolic groups of Gromov (see [10], [8], [3]).

In this context, A. Casson has invented a certain geometrical condition for $\pi_1 M^3$, the so-called condition $\hat{C}_\alpha$ and has shown that if $\pi_1 M^3$ verifies $\hat{C}_\alpha$ for *very small values of* α (something like $\alpha \leq 2$), then $\pi_1^\infty \tilde{M}^3 = 0$ (see [23], [6], [22]). It is also conjectured that for an arbitrary M^3, the fundamental group $\pi_1 M^3$ always verifies $\hat{C}_\alpha$ for sufficiently big α. Recent results of S. Gersten show that this is indeed the case for very large classes of 3-manifold [6], [7].

In the present paper, which uses the methods of [16] and [12], we show that if $\pi_1 M^3$ verifies Casson's condition $\hat{C}_\alpha$ for *some* α, and also a certain additional "very mild" requirement, then indeed $\pi_1^\infty \tilde{M}^3 = 0$. In precise form this is the theorem below (Casson's $\hat{C}_\alpha$ is the 1° from the theorem and 2° is the extra, mild, requirement). Some prerequisites will be necessary before we can state our theorem.

Let π be a finitely generated group and $A = \{\gamma_1^{\pm 1}, \ldots, \gamma_m^{\pm 1}\}$ a system of generators to which the Cayley graph $\Gamma = \Gamma(\pi, A)$ is associated. We introduce on π the norm $\|x\|_A$ induced by A and the distance function $d_A(x, y) = \|x^{-1}y\|_A$. In the Cayley graph Γ we consider the sphere of radius n and the ball of radius n, defined respectively by

$$S(n) \underset{\text{def}}{=} \{x \in \pi \text{ such that } \|x\|_A = n\} \tag{1.1}$$

and

$$B(n) \underset{\text{def}}{=} \{x \in \pi \text{ such that } \|x\|_A \leq n\}.$$

Assume now that π is finitely presented, so that to the finite system of generators A we can add a finite system of relators $R = (R_1, \ldots, R_r)$; it will be understood that R is closed under the operations of taking inverses and also under cyclic permutations of the letters in A expressing the words R_i. By definition an (A, R)-*homotopy* between two paths l', l'' of Γ with the same end-points x, y is a finite sequence of paths $l_0 = l', l_1, l_2, \ldots, l_{N-1}, l_N = l''$

with the same endpoints as l', l'' and such that for every $i = 1, \ldots, N$ the following conditions hold.

(a) There are elements $x_i, y_i \in \pi$ and decompositions (as products of paths in $\Gamma(\pi, A)$)

$$l_{i-1} = a_i \cdot b_{i-1}(i) \cdot c_i, \quad l_i = a_i \cdot b_i(i) \cdot c_i, \tag{1.2}$$

where a_i goes from x to x_i, $b_{i-1}(i)$ and $b_i(i)$ go from x_i to y_i, while c_i goes from y_i to y.

(b) The element of π expressed by the word $b_i(i) \cdot b_{i-1}(i)^{-1}$ is in R (see Fig. 1)

We can state now the main result of this paper.

THEOREM. *Let M^3 be a closed 3-manifold which is such that $\pi = \pi_1 M^3$ has the following two properties*

1°. *There is a finite presentation $A = \{\gamma_1^{\pm 1}, \ldots, \gamma_m^{\pm 1}\}$, $R = (R_1, \ldots, R_r)$ for $\pi_1 M^3$ and two constants α, η such that for any two paths l', l'' of $\Gamma(\pi_1 M^3, A)$ with $\|l'\|_A, \|l''\|_A \le n$ and with the same endpoints we can find an R-homotopy*

$$l_0 = l', l_1, l_2, \ldots, l_N = l'' \tag{1.3}$$

satisfying the estimate

$$\|l_i\|_A \le \alpha n + \eta \tag{1.4}$$

for each i. (We denote here by $\|l\|$ the length of the path l).

2°. *There is a second finite system of generators $B = \{g_1^{\pm 1}, \ldots, g_q^{\pm 1}\}$ of $\pi_1 M^3$, for which we can find non-negative constants $C, \varepsilon > 0$, and C' such that whenever we have two elements of $\pi_1 M^3$ belonging to the sphere of radius n of $\Gamma(\pi_1 M^3, B)$, $x, y \in S(n) \subset \Gamma(\pi_1 M^3, B)$ which are such that $d_B(x, y) \le 3$, then we can join x to y inside the ball of radius n, $B(n) \subset \Gamma(\pi_1 M^3, B)$ by a path L with*

$$\|L\|_B \le C n^{1-\varepsilon} + C'. \tag{1.5}$$

Under these two conditions, if K is any compact subset of the universal covering space $\tilde{M}^3$, we can always find a smooth bounded COMPACT, SIMPLY-CONNECTED submanifold $U^3 \subset \tilde{M}^3$, containing K. In other words, $\tilde{M}^3$ is simply-connected at infinity.

Remarks. (A). Concerning condition 2° of our theorem, notice that for an arbitrary finitely generated group G and an arbitrary system of generators B, *ANY* pair of points $x, y \in S(n) \subset \Gamma(G, B)$ can be joined inside $B(n)$ by a path L with

$$\|L\|_B \le 2n. \tag{1.6}$$

This remark certainly makes our (1.5) (imposed only for $x, y \in G$ with $d_B(x, y)$ sufficiently small) look like a very mild restriction, at least of first sight. How mild it actually is, I do not know. Clearly Cannon's almost convexity implies our condition 2°. So, fundamental groups

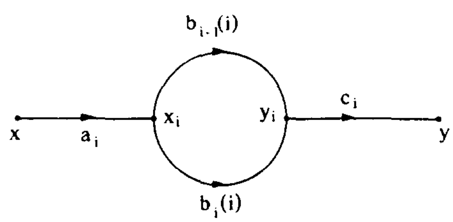

Fig. 1. The closed loop here is a relator in R.

of geometric 3-manifolds of type S^3, R^3, H^3, $H^2 \times R$, $S^2 \times R$ and NIL, which are known to be almost convex [2], [20] as well as most PSL groups [21], satisfy our condition 2°. L. Funar has shown that at least for some SOL groups, condition 2° is not fulfilled, for some natural systems of generators B [5].

(B) This leads to the more philosophical question whether it is possible to find a geometrical condition for $\pi_1 M^3$ general enough so that it holds for all M^3's and at the same time strong enough so that it implies that $\pi_1^\infty \tilde{M}^3 = 0$. As a possible substitute to such a program, I offer the following

Conjecture. If M^3 has infinite π_1, and if n is sufficiently large, then there is a closed subset $F \subset \tilde{M}^3$ such that F is a countable union of two-by-two disjoint tame Cantor sets contained in $\tilde{M}^3$ and that $(\tilde{M}^3 - F) \times (0) \subset (\tilde{M}^3 - F) \times D^n$ can be engulfed inside a PROPER codimension zero submanifold $U \subset (\tilde{M}^3 - F) \times D^n$ which has NO HANDLES OF INDEX ONE.

Together with an easy extension of [14], this would certainly settle the conjecture $\pi_1^\infty \tilde{M}^3 = 0$ in whole generality. A first step in the direction of the CONJECTURE above is taken in [17], [18].

2. SOME PRELIMINARIES

In this section, we will review some definitions and constructions from [13], [15], [14].

If $A \xrightarrow{F} B$ is any map, we will define $M_2(F) \subset A$ by

$$M_2(F) = \{x \in A \text{ such that } \operatorname{card} F^1 F(x) > 1\}$$

and by $M^2(F) \subset A \times A$ the lift of $M_2(F)$ to $A \times A$, i.e. $M_2(F) = \{(x, y) \in A \times A \mid x \neq y, f(x) = f(y)\}$.

One of the ingredients for our Theorem is the following

DEHN-TYPE LEMMA. *Let X and Y be two simply-connected 3-manifolds. We assume X to be compact, connected, with $\partial X \neq \emptyset$ and Y to be open.*

We are given a commutative diagram

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[dr, "f"'] & \operatorname{int} X \subset X \arrow[d, "F"] \\ & Y \end{tikzcd}$$

where K is a compact connected set, g and f are embeddings and F is a smooth generic immersion. If the following condition is also fulfilled

$$(gK) \cap M_2(F) = \emptyset$$

then $fK \subset Y$ is contained inside a compact simply connected smooth 3-dimensional submanifold $N \subset Y$.

The next item will be *double point structures*. Let P be a (not necessarily locally finite) 3-dimensional simplicial complex, M^3 a 3-manifold and $P \xrightarrow{f} M^3$ a non-degenerate simplicial map (i.e. if $\sigma \in P$ is a simplex of P, then $\dim f\sigma = \dim \sigma$). We will denote by $\Phi(f) \subset P \times P$ the subset $\Phi(f) = M^2(f) \cup (\operatorname{diag} P)$; in other terms $\Phi(f)$ *is the equivalence relation* on P defined by

$$(x, y) \in \Phi(f) \Leftrightarrow fx = fy.$$

By definition, $\mathrm{Sing}(f) \subset P$ is the subcomplex whose points $z \in P$ are such that $f \mid \mathrm{Star}(z)$ is *non immersive*; in other words $z \in \mathrm{Sing}(f)$ iff there are two *distinct* simplexes $\sigma_1, \sigma_2 \subset P$ with $z \in \sigma_1 \cap \sigma_2$ are $f(\sigma_1) = f(\sigma_2)$. Clearly the quotient space $P/\Phi(f)$ is isomorphic to the image fP. We are also interested in equivalence relations $R \subset \Phi(f)$ which are such that if $x \in \sigma_1$, $y \in \sigma_2$, where σ_1 and σ_2 are two simplexes of P of the same dimension with $fx = fy$ and $f(\sigma_1) = f(\sigma_2)$, then

$$(x, y) \in R \Rightarrow \{R \text{ identifies } \sigma_1 \text{ to } \sigma_2\}.$$

Such equivalence relations automatically have the property that P/R is a simplicial complex and the induced map $P/R \to M^3$ is also simplicial. Among these R's, there is a particularly interesting equivalence relation $\Psi(f) \subset \Phi(f)$, the basic features of which are summarized in the following

LEMMA 2.1 (I). *There exists an equivalence relation $\Psi(f) \subset \Phi(f)$ which is completely characterized by the following two properties* (Ia *and* Ib).

(Ia) *If we consider the natural commutative diagram*

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & M^3 \\ & P/\Psi(f) \arrow[ur, "f_1"'] & \end{tikzcd} \tag{2.1}$$

then $\mathrm{Sing}(f_1) = \emptyset$ i.e. f_1 is an immersion.

(Ib) *There is no $R \subset \Phi(f)$, smaller than $\Psi(f)$ having this property. In other words, $\Psi(f)$ is the smallest equivalence relation, compatible with f which kills all the singularities.* (*We will also say that "$\Psi(f)$ is the equivalence relation which is commanded by the singularities of f".*)

(II) *The canonical map $\pi_1 P \xrightarrow{\pi_*} \pi_1(P/\Psi(f))$ is surjective. In particular, if P is simply connected, then so is $P/\Psi(f)$.*

These facts are proved in [13].

We will need to apply the material about double point structures sketched before to a specific situation. This will involve a relatively unusual "naive" view of universal covering spaces (for closed 3-manifolds but which actually can be extended to much more general situations). I start by recalling the following standard way for representing a closed 3-manifold M.

I start with a polyhedral 3-ball Δ with triangulated $\partial\Delta$, containing an even number of triangles $h_1, h_2, \ldots, h_{2p}$. We are also given a fixed point free involution

$$S \underset{\text{def}}{=} \{h_1, h_2, \ldots, h_{2p}\} \xrightarrow{j} S,$$

and M^3 is the quotient space Δ/ρ, where the equivalence relation ρ identifies each h_s to jh_s, by an appropriate linear isomorphism. We will consider the free monoid $\mathscr{G}$ generated by S and 1. We also consider the space T obtained from the disjoint union $\sum_{x \in \mathscr{G}} x\Delta$ by glueing, for each $x \in \mathscr{G}$ and $h_s \in S$, the fundamental domains $x\Delta$ and $(xh_s)\Delta$ along their respective h_s and jh_s faces, in a Cayley graph manner. We do *not* restrict ourselves here to reduced words x, which makes T quite complicated already at the local level; "reduced" means here that any occurance $jh_i \cdot h_i$, $h_i \cdot jh_i$ is cancelled. There is an obvious tautological map $T \xrightarrow{f} M^3$ which sends each fundamental domain $x\Delta \subset T$ *identically* onto $\Delta \to M^3$. This map just unrolls undefinitely the fundamental domain $\Delta \to M^3$, along its faces, like the developing map [25], [24]. In section 2) of [15], it is proved the following.

LEMMA 2.2. *The canonical map* (see (2.1))

$$T/\Psi(f) \xrightarrow{f_1} M^3$$

is the universal covering space of M^3.

We choose now, once for all, a fundamental domain $\{\bar{g}_1, \ldots, \bar{g}_p\} \subset S$ for the action of j on S. This fundamental domain induces a system of generators for $\pi_1 M^3$: we choose as base-point the centre $* \in \Delta$ and we associate to each $\bar{g}_i$ the closed loop of M^3 which, in Δ, joins the center of $(j\bar{g}_i)$ to the center of $\bar{g}_i$. Call $g_i \in \pi_1 M^3 = \pi_1(M^3, *)$ the corresponding element. With this, there is an obvious surjective morphism (in the category of semi-groups)

$$\bar{\mathscr{G}} \xrightarrow[\chi]{} \pi_1 M^3,$$

which is such that $\chi(\bar{g}_i) = g_i$, $\chi(j\bar{g}_i) = g_i^{-1}$. Generically, we will denote $\chi(\bar{g})$ by g.

In [16] the following lemma is proved; it will be useful in this paper too.

LEMMA 2.3. *For any finite system of elements* $\{\gamma_1^{\pm 1}, \ldots, \gamma_s^{\pm 1}\} \subset \pi_1 M^3$, *we can choose a representation* $\Delta/\rho = M^3$ *such that for the* $\{g_1^{\pm 1}, \ldots, g_p^{\pm 1}\} \subset \pi_1 M^3$, *obtained by the construction above, we have*

$$\{\gamma_1^{\pm 1}, \ldots, \gamma_s^{\pm 1}\} \subset \{g_1^{\pm 1}, \ldots, g_p^{\pm 1}\}.$$

3. STRATEGY OF THE PROOF

We consider the set of generators $B = \{g_1^{\pm 1}, \ldots, g_q^{\pm 1}\}$ from point 2° in our theorem and the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$. Whenever the contrary is not explicitly stated, all the norm $\| \ldots \|$ or distances $d(\ldots, \ldots)$ will be computed with respect to B (i.e. they will be $\| \ldots \|_B$ and $d_B(\ldots, \ldots)$, respectively).

Like in the previous section, we will represent M^3 as the quotient of some fundamental domain Δ. The set of triangles of $\partial\Delta$ is $S = \{h_1, h_2, \ldots, h_{2p}\}$ and we choose a fundamental domain $\{\bar{g}_1, \ldots, \bar{g}_p\} \subset S$ for the fixed-point free involution $S \xrightarrow{j} S$. As already explained in the previous section, we can associate a $g_i \underset{\text{def}}{=} \chi(\bar{g}_i) \in \pi_1 M^3$ to $\bar{g}_i$ (and correspondingly g_i^{-1} to $j\bar{g}_i$). Lemma 2.3 tells us that we can assume without any loss of generality that $B = \{g_1^{\pm 1}, \ldots, g_q^{\pm 1}\}$ for some $q \leq p$. We have canonical map

$$B \to \bar{\mathscr{G}}, \tag{3.0}$$

which sends, for $i \leq q$, g_i into $\bar{g}_i \in S$ and g_i^{-1} into $j\bar{g}_i \in S$. This is a section of the morphism

$$\bar{\mathscr{G}} \xrightarrow[\chi]{} \pi_1 M^3.$$

[*Reminder*. All the norms $\|g\|$ for $g \in \pi_1 M^3$ in the discussion which follows, will be computed with respect to B *and* never with respect to the larger system of generators $B = \{g_1^{\pm 1}, \ldots, g_p^{\pm 1}\}$].

We will chosse once for all a lift of $\Delta \to M^3$ to $\tilde{M}^3$

$$\begin{tikzcd} & \tilde{M}^3 \arrow[d, "\pi"] \\ \Delta \arrow[ur, "\Delta_0"] \arrow[r] & M^3 \end{tikzcd} \tag{3.1}$$

and the image of Δ_0 will be denoted again by Δ so that Δ is now a fundamental domain for the action of $\pi_1 M^3$ on $\tilde{M}^3$. Once $\Delta \xrightarrow{\Delta_0} \tilde{M}^3$ is fixed by (3.1), we have an obvious commutative diagram, with (T,f) like in Section 2

$$\begin{tikzcd} T \arrow[rr, "F = F(\Delta_0)"] \arrow[dr, "f"'] & & \tilde{M}^3 \arrow[dl, "\pi"] \\ & M^3 & \end{tikzcd} \tag{3.2}$$

where F sends $\bar{g}\Delta$ onto the fundamental domain $g\Delta \to \tilde{M}^3$, with $g = \chi(\bar{g})$.

LEMMA 3.1. *We have* $\Psi(F) = \Phi(F)$.

Proof. Since π is a covering projection, the singularities of f and F have to be killed by the same foldings (2.2), (2, 3), ... so that $\Psi(f) = \Psi(F) \subset \Phi(F)$. We know, on the other hand (see Lemma 2.2), that $T/\Psi(f) = \tilde{M}^3$, which is also equal to $T/\Phi(F)$, and this implies our result.

I will denote by $T_n \subset T = \bigcup_{\bar{x} \in \bar{\mathscr{G}}} \bar{x}\Delta$ the compact subspace obtained by restricting ourselves to those $\bar{x}$'s which, expressed as a word in $\{h_1, h_2, \ldots, h_{2p}\}$, have length $\leq n$. This space can be obtained by a sequence of Whitehead dilations, starting from a point, i.e. it is collapsible. I will use the relations $\Psi_n = \Psi(F \mid T_n)$ and $\Phi_n = \Phi(F) \mid T_n = \Phi(F) \mid T_n$. In general, we only have $\Psi_n \subset \Psi(F) \mid T_n$, but we also have the following important consequence of Lemma 3.1.

LEMMA 3.2. *For each* $M \in Z^+$, *there is an* $\bar{M} = \bar{M}(M) \in Z^+$ *with* $\bar{M} \geq M$, *such that* $\Psi_{\bar{M}} \mid T_M = \Phi_M$.

Proof. Lemma 3.1 tells us that $\Psi(F) \mid T_M = \Phi_M$, which is the kind of thing we want to prove, but with $\bar{M} = \infty$. On the other hand, $\Psi(F)$ can be exhausted by a sequence of folding maps modelled on the first transfinite ordinal ω (see (c) in section 2 above). Since only finitely many of these folding maps involve T_M, we have our result.

Let us stop now for a minute and review the properties of the map $T \xrightarrow{F} \tilde{M}^3$. The space T is a tree-like union of fundamental domains Δ (in particular it can be gotten from a point by Whitehead dilatations), we have $\Psi(F) = \Phi(F)$, but (unfortunately!) any compact subspace $K \subset \tilde{M}^3$ is touched by the images of INFINITELY many fundamental domains in T.

Once the compact $K \subset \tilde{M}^3$ is given, our STRATEGY for proving our theorem will be to construct a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "G"'] & & T \arrow[dl, "F"] \\ & \tilde{M}^3 & \end{tikzcd} \tag{3.3}$$

with the following list of properties.

(I) Like T, the space Z is a tree-like union of fundamental domains Δ and g is a non-degenerate "simplicial" map, sending fundamental domains of Z isomorphically into fundamental domains of T. (For example, Z could be a sub-tree of fundamental domains of T).

(II) The arrow G is also a non-degenerate surjective simplicial map, just like F, AND $\Psi(G) = \Phi(G)$.

(III) Properties I and II above are, as already said, shared by (T, F) too. But for $Z \xrightarrow{G} \tilde{M}^3$ we ask that the given compact set $K \subset \tilde{M}^3$ should be touched only by FINITELY many images of fundamental domains of Z.

LEMMA 3.3. *If the given compact K is connected and if we can construct a map $Z \xrightarrow{G} \tilde{M}^3$ like in diagram* (3.3) *and with properties* I, II, III *above, then our $K \subset \tilde{M}^3$ can be engulfed inside a bounded simply-connected* 3*-dimensional submanifold $N^3 \subset \tilde{M}^3$; in particular we will have $\pi_1^\infty \tilde{M}^3 = 0$.*

Proof. The proof involves the following steps.

Step I. Consider any exhaustion of Z by *collapsible finite* unions of fundamental domains

$$Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots \subset Z.$$

It follows from our property III that there is an Y_n such that

$$G(Z - Y_n) \cap K = \varnothing.$$

Since $Z/\Phi(G) = \tilde{M}^3$ this implies, among other things, that K can be lifted to $Y_n/\Phi(G \mid Y_n)$.

Step II. I claim that given our n we can find an $m > n$ such that

$$\Psi(G \mid Y_m) \mid Y_n = \Phi(G \mid Y_n). \tag{3.4}$$

Once we know that $\Psi(G) = \Phi(G)$, this fact can be proved exactly like in Lemma 3.2 (where we have made use of $\Psi(F) = \Phi(F)$ in a similar context).

Step III. We consider the inclusion map

$$K \subset Y_n/\Phi(G \mid Y_n) \subset Y_m/\Psi(G \mid Y_m)$$

and the obvious commutative diagram

$$\begin{tikzcd} K \arrow[r, hook, "i"] \arrow[dr] & Y_m/\Psi(G \mid Y_m) \arrow[d, "g_1 = \text{immersion}"] \\ & \tilde{M}^3 \end{tikzcd}$$

This diagram has the following properties:

III.1 $Y_m/\Psi(G \mid Y_m)$ is a *simply-connected* finite 3-dimensional polyhedron (see II in Lemma 2.1).

III.2 The map g_1 is an immersion (see Ia in Lemma 2.1).

III.3 If we denote by $M_2(g_1) \subset Y_m/\Psi(G \mid Y_m)$ the set of double points of g_1, then

$$\boxed{K \cap M_2(g_1) = \varnothing}$$

If we replace $Y_m/\Psi(G \mid Y_m)$ with a very thin 3-dimensional regular neighbourhood, compatible with g_1, then we are exactly in the conditions of our *Dehn-type* lemma, which assures us now of an engulfing $K \subset N^3 \in \tilde{M}^3$ with N^3 compact and simply-connected as desired.

Now since any compact $K_1 \subset \tilde{M}^3$ is contained inside a connected compact $K \subset \tilde{M}^3$, the only thing left in order to prove our Theorem is to exhibit a $Z \xrightarrow{G} \tilde{M}^3$ with all the desired properties for an arbitrary given compact connected subset K of $\tilde{M}^3$.

4. CONSTRUCTION OF $Z \xrightarrow{G} \tilde{M}^3$

We start by reviewing the corresponding construction from [16]; after the space Z_1 (see below) has been introduced, the two constructions differ.

We start by noticing that, for every $\bar{g} \in \bar{\mathscr{G}}$, there is a map $T \xrightarrow{\bar{g}} T$ which sends each $x\Delta \subset T$ onto $\bar{g}x\Delta \subset T$. The map $\bar{g}$ is clearly compatible with the incidence relations of T, and it is an isomorphism between T and $\bar{g}T \subset T$; this is not a surjection onto T (since non-reduced words are used). We also have commutative diagram which connects it to the left action $\pi_1 M^3 \times \tilde{M}^3 \to \tilde{M}^3$

$$\begin{array}{ccc} T & \xrightarrow{\bar{g}} & T \\ {\scriptstyle F = F(\Delta_0)} \downarrow & & \downarrow {\scriptstyle F = F(\Delta_0)} \\ \tilde{M}^3 & \xrightarrow{g = \chi(\bar{g})} & \tilde{M}^3 \end{array}$$

For any $g \in \pi_1 M^3$, we consider all the *geodesic paths* of the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$ joining $1 \in \Gamma$ to $g \in \Gamma$. For a given g, there are only *finitely many* such paths and they take the general form

$$\alpha_i(g) = (1, g_{j(1)}, g_{j(1)}g_{j(2)}, \ldots, g_{j(1)}g_{j(2)} \cdots g_{j(n)} = g), \tag{4.1}$$

where $n = \|g\|$, $g_{j(k)} \in B$ and $i = 1, 2, \ldots, \rho(g) < \infty$. By using the canonical map $B \to \bar{\mathscr{G}}$ (see (3.0)), we have a canonical lift of $\alpha_i(g)$ to $\bar{\mathscr{G}}$

$$\bar{\alpha}_i(g) = (1, \bar{g}_{j(1)}, \bar{g}_{j(1)}\bar{g}_{j(2)}, \ldots, \bar{g}_{j(1)}\bar{g}_{j(2)} \cdots \bar{g}_{j(n)} \underset{\text{def}}{=} \bar{g}), \tag{4.2}$$

To (4.2), we can associate a continuous path of fundamental domains in T

$$\bar{\alpha}_i(g)\Delta \underset{\text{def}}{=} 1 \cdot \Delta \cup \bar{g}_{j(1)}\Delta \cup \bar{g}_{j(1)}\bar{g}_{j(2)}\Delta \cup \ldots \cup \bar{g}\Delta \subset T. \tag{4.3}$$

We consider the quotient space Z_1 of the disjoint union

$$\sum_{g \in \pi_1 M^3;\ i = 1, \ldots, \rho(g)} \bar{\alpha}_i(g)\Delta,$$

obtained by identifying *all* $1 \cdot \Delta \subset \bar{\alpha}_i(g)\Delta$ together; so Z_1 is locally finite, except at $1 \cdot \Delta$. The fundamental domains $\bar{g}\Delta \subset \bar{\alpha}_i(g)\Delta \subset Z_1$ which are end-points of the corresponding paths $\bar{\alpha}_i(g)\Delta$ will be, by definition, ***red fundamental domains***. From here on, the construction of (Z, G) proceeds as follows.

Remembering that our construction of Z will depend on an initially given compact connected subset $K \subset \tilde{M}^3$ we will choose now an $R > 0$ large enough so that if for some $g \in \pi_1 M^3$ we have $g\Delta \cap K \neq \emptyset$, then $\|g\| < R$. Let us consider now some positive integer $r \leq \|g\|$; we will denote by $\alpha_i(g) \mid r$, $\bar{\alpha}_i(g) \mid r$ and $\bar{\alpha}_i(g)\Delta \mid r$ the obvious truncations of (4.1), (4.2) and (4.3), respectively. We will also define a quotient-space $Z_2 = Z_2(R)$ of Z_1 as follows. Any time we find $g_1, g_2 \in \pi_1 M^3$, $i_1 \leq \rho(g_1)$, $i_2 \leq \rho(g_2)$ and an $r \leq \inf(R, \|g_1\|, \|g_2\|)$ such that $\alpha_{i_1}(g_1) \mid r = \alpha_{i_2}(g_2) \mid r$, we identify $(\bar{\alpha}_{i_1}(g_1)\Delta) \mid r$ to $(\bar{\alpha}_{i_2}(g_2)\Delta) \mid r$ in the obvious manner. So Z_2 is locally finite except along the sphere of radius R. A fundamental domain of Z_2 which is the image of a red fundamental domain of Z_1 will be, by definition, ***red***.

Both Z_1 and Z_2 are part of an obvious commutative diagram, analogous to (3.3)

$$\begin{array}{ccc} Z_\varepsilon & \xrightarrow{g_\varepsilon} & T \\ & {\scriptstyle G_\varepsilon} \searrow \quad \swarrow {\scriptstyle F} & \\ & \tilde{M}^3 & \end{array} \tag{4.4.ε}$$

with $\varepsilon = 1, 2$. Both these objects Z_1 and Z_2 verify property I of our STRATEGY and $Z_2 \xrightarrow{G_2} \tilde{M}^3$ also verifies III. But in order to fulfil II (i.e. "double points should be commanded by singularities") we will need to enlarge our Z_2. This will be done in two successive stages, again. Now point 2° in our theorem provides us with a quantity $\varepsilon > 0$, and

I will consider three more positive parameters $C_1, C_2, M \in Z^+$. Depending on them I will construct an object which for sufficiently large C_1, C_2, M (how large exactly will be made precise later) will be our desired Z. But for the moment we will NOT specify the sizes of C_1, C_2, M.

For any $n \in Z^+$ I will consider the set $P(n)$ of all continuous paths of fundamental domains of T, starting at $1 \cdot \Delta$ and of length $\leq C_1 n^{1-\varepsilon} + C_2$; here $\varepsilon > 0$ is like in point 2° from our Theorem. So a typical element $\lambda \in P(n)$ takes the form

$$\lambda = 1 \cdot \Delta \cup h_{i_1}\Delta \cup h_{i_1}h_{i_2}\Delta \cup \ldots \cup h_{i_1}h_{i_2}\ldots h_{i_m}\Delta \tag{4.5}$$

with $m \leq C_1 n^{1-\varepsilon} + C_2$ and with consecutive fundamental domains (4.5) glued together in the obvious fashion (i.e. like in T). With this we define

$$Z_3(C_1, C_2) = Z_2 + \underbrace{\sum}_{\bar{x}\Delta,\, \lambda} \bar{x}\lambda, \tag{4.6}$$

where $\bar{x}\Delta \in \{$the set of RED fundamental domains of $Z_2\}$, where for a given RED $\bar{x}\Delta$, we take all the $\lambda \in P(\|x\|_B)$ (here $x = \chi(\bar{x}) \in \pi_1 M^3$) and where the $\bar{x}\Delta$ appearing in

$$\bar{x}\lambda = \bar{x}\Delta \cup \bar{x}h_{i_1}\Delta \cup \bar{x}h_{i_1}h_{i_2}\Delta \cup \ldots \cup \bar{x}h_{i_1}h_{i_2}\ldots h_{i_m}\Delta$$

is identified to the corresponding RED fundamental domain $\bar{x}\Delta \subset Z_2$.

[*Notational convention.* If $\bar{x} \in \bar{\mathfrak{G}}$ we will denote by x the image of $\bar{\mathfrak{G}}$ in $\pi_1 M^3$ via the natural map $\bar{\mathfrak{G}} \xrightarrow{x} \pi_1 M^3$].

Now, for our parameter $M \in Z^+$, Lemma 3.2 gives us another quantity $\bar{M} = \bar{M}(M) > M$ and with this we set

$$Z_4(C_1, C_2, M) \underset{\text{def}}{=} Z_3(C_1, C_2) + \sum_{\bar{y}\Delta} \bar{y}T_{\bar{M}} \tag{4.7}$$

where $\bar{y}\Delta$ is an arbitrary fundamental domain appearing in the piece $\underbrace{\sum}_{\bar{x}\Delta,\, \lambda} \bar{x}\lambda \subset Z_3(C_1, C_2)$ (see 4.6) and where $\bar{y}\Delta \subset \bar{y}T_{\bar{M}}$ is always identified to $\bar{y}\Delta \subset Z_3(C_1, C_2)$.

There is an obvious commutative diagram, analogous to (3.3)

$$\begin{tikzcd} Z_4(C_1, C_2, M) \arrow[rr, "g = g(C_1, C_2, M)"] \arrow[dr, "G = G(C_1, C_2, M)"'] & & T \arrow[dl, "F"] \\ & \tilde{M}^3 & \end{tikzcd} \tag{4.8}$$

which clearly has property I of our STRATEGY. The next lemma tells us that it also has property III.

LEMMA 4.1. *The map $Z_4(C_1, C_2, M) \xrightarrow{G} \tilde{M}^3 \supset K$ has the property that only FINITELY many fundamental domains of $Z_4(C_1, C_2, M)$ touch K.*

Proof. We know that this is already true for the part $Z_2 \subset Z_4(C_1, C_2, M)$. We go from Z_2 to Z_4 by adding for each RED $\bar{x}\Delta \subset Z_2$ the finite arborescent contribution, based at $\bar{x}\Delta$ (see (4.6), (4.7))

$$\sum_{\lambda} \bar{x}\lambda + \underbrace{\sum}_{\bar{y}\Delta \subset \bar{x}\lambda} \bar{y}T_{\bar{M}} \tag{4.9}$$

which we will denote by $t(\bar{x}\Delta) \subset Z_4$. If $\|x\| = n$ then for an arbitrary $\bar{z}\Delta \subset t(\bar{x}\Delta)$ we have the estimate

$$d_B(x, z) \leq C_1 n^{1-\varepsilon} + C_2 + \bar{M}. \tag{4.10}$$

But for given n there are only FINITELY many RED fundamental domains $\bar{x}\Delta \subset T_2^\infty$ with $\|x\|_B \le n$, and since $\lim_{n\to\infty} (n - (C_1 n^{1-\varepsilon} + C_2 + \bar{M})) = \infty$, only FINITELY many $\bar{z}\Delta$'s (appearing in ALL the $t(\bar{x}\Delta)$'s possible) have images which can touch K. [We use here the following fact (see [Mi]). Let us endow M^3 with an arbitrary Riemanian metric which we lift, afterwards, to $\tilde{M}^3$. If $p \in \tilde{M}^3$ and $g \in \pi_1 M^3$ then, with respect to this last metric, we have

$$\lim_{\|g\|\to\infty} d(p, g\cdot p) = \infty;$$

moreover this estimate is uniform when p moves inside a compact subset of $\tilde{M}^3$].

5. THE PROPERTY $\Phi(G) = \Psi(G)$

It remains to show that we can choose C_1, C_2, M so as to fulfil the requirement II from our STRATEGY.

I will start by introducing the following definition. Let λ', λ'' be two paths of $\Gamma = \Gamma(\pi_1 M^3, B)$ with the same endpoints x, y. By definition a *B-homotopy* connecting λ' to λ'' is a finite sequence of paths of Γ, with the same endpoints x, y as before

$$\lambda' = \lambda_0, \lambda_1, \ldots, \lambda_N = \lambda'' \tag{5.1}$$

such that for any $i = 1, \ldots, N$ there are elements $x_i, y_i \in \pi_1 M^3$ and also a decomposition (as product of paths in $\Gamma = \Gamma(\pi_1 M^3, B)$)

$$\lambda_{i-1} = \alpha_i \cdot \beta_{i-1}(i) \cdot \gamma_i, \quad \lambda_i = \alpha_i \cdot \beta_i(i) \cdot \gamma_i, \tag{5.2}$$

where α_i goes from x to x_i, $\beta_{i-1}(i)$ and $\beta_i(i)$ go from x_i to y_i and γ_i goes from y_i to y.

LEMMA 5.1. *There exists three constants $\tilde{C}_1, \tilde{C}_2, \tilde{M}$ such that if λ', λ'' are two paths of $\Gamma(\pi_1 M^3, B)$ with the same endpoints and with $\|\lambda'\|_B, \|\lambda''\|_B \le n$ then we can join them by a B-homotopy such that for each i we have*

$$\|\lambda_i\|_B \le \tilde{C}_1 n + \tilde{C}_2, \tag{5.3}$$

and

$$\|\textit{the closed path } \beta_i(i)\cdot\beta_{i-1}(i)^{-1}\|_B \le \tilde{M}. \tag{5.4}$$

Proof. This will follow, of course, from the property 1° in our Theorem, which holds for the presentation (A, R) of $\pi_1 M^3$. We have two bases for $\pi_1 M^3$, $A = \{\gamma_1^{\pm 1}, \ldots, \gamma_m^{\pm 1}\}$ and $B = \{g_1^{\pm 1}, \ldots, g_q^{\pm 1}\}$; for simplicity's sake we will stop writing the exponents ± 1. We choose once for all expressions of the g_i's in terms of the γ's and of the γ_i's in terms of the g's

$$g_i = g_i(\gamma_1, \gamma_2, \ldots, \gamma_m), \qquad \gamma_j = \gamma_j(g_1, g_2, \ldots, g_q). \tag{5.5}$$

For each $j = 1, \ldots, m$ we can read the expression $\gamma_j(g_1, g_2, \ldots, g_q)$ as a path $\tilde{\gamma}_j$ of $\Gamma(\pi_1 M^3, B)$ going from 1 to $\gamma_j \in \pi_1 M^3$. Similarly, for each $i = 1, \ldots, q$ we can read the expression

$$g_i(\gamma_1(g_1, g_2, \ldots, g_q), \gamma_2(g_1, g_2, \ldots, g_q), \ldots, \gamma_m(g_1, g_2, \ldots, g_q))$$

as a path of $\Gamma = \Gamma(\pi_1 M^3, B)$ going from 1 to g_i; this path, which is an obvious composition of $\tilde{\gamma}_j$'s will be denoted by $\tilde{g}_i$. We will describe now our B-homotopy in three stages.

Stage 1°. Let us say that

$$\lambda'_0 \underset{\text{def}}{=} \lambda' = (x, xg_{i_1}, xg_{i_1}g_{i_2}, \ldots, xg_{i_1}g_{i_2}\cdots g_{i_n} = y)$$

and consider to begin with

$\lambda_1' = \{\lambda_0'$ with the segment $[x, xg_{i_1}]$ replaced by the path $x \cdot \tilde{g}_{i_1}$ (which also goes from x to xg_{i_1} in $\Gamma = \Gamma(\pi_1 M^3, B))\}$,

$\lambda_2' = \{\lambda_1'$ with the segment $[xg_{i_1}, xg_{i_1}g_{i_2}]$ replaced by the path $xg_{i_1} \cdot \tilde{g}_{i_2}\}$,

....

$\lambda_n' = \{\lambda_0'$ with EVERY segment $[xg_{i_1} \ldots g_{i_l}, xg_{i_1} \ldots g_{i_l}g_{i_{l+1}}]$ replaced by the path $xg_{i_1} \ldots g_{i_l} \cdot \tilde{g}_{i_{l+1}}\}$.

This is a B-homotopy connecting λ_0 to λ_n and if we denote

$$a_1 \underset{\text{def}}{=} \sup_i \|g_i\|_A, \qquad b_1 \underset{\text{def}}{=} \sup_j \|\gamma_j\|_B,$$

then for $i \le n$ we have $\|\lambda_i'\|_B \le na_1 b_1$ while each of the closed paths via which λ_{l+1}' differs from λ_l' is of B-length $\le a_1 b_1 + 1$ (see Fig. 2).

Stage 2°. In a completely similar fashion we can construct a B-homotopy, with exactly the same estimates as for λ_t', but starting with $\lambda_0'' \underset{\text{def}}{=} \lambda''$, let us call it $\lambda_0'', \lambda_1'', \ldots, \lambda_n''$.

Each of the paths λ_n', λ_n'' of $\Gamma = \Gamma(\pi_1 M^3, B)$ joins x to y and is a product of not more than $a_1 n$ "elementary" paths $\tilde{\gamma}_1 \tilde{\gamma}_2, \ldots, \tilde{\gamma}_m$; let us say that $\lambda_n' = \lambda_n'(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$, $\lambda_n'' = \lambda_n''(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$.

Stage 3°. It is now that we have to use point 1° from our Theorem. Since each of the λ_n', λ_n'' is a product of paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ we can also read them as paths in the Cayley graph $\Gamma(\pi_1 M^3, A)$ joining x to y, namely $l' = \lambda_n'(\gamma_1, \ldots, \gamma_m)$, $l'' = \lambda_n''(\gamma_1, \ldots, \gamma_m)$. We have $\|l'\|_A$, $\|l''\|_B \le na_1$. Point 1° from our Theorem tells us that there exists an (A, R)-homotopy of paths l_t of the Cayley graph $\Gamma(\pi_1 M^3, A)$ connecting l' to l'' and having the following two properties: (i) each l_{t+1} differs from l_t exactly by a relator $R_t \in R$; (ii) for each t we have an estimate $\|l_t\|_A \le \alpha a_1 n + \eta$.

Each R_t can also be read as a closed path $\tilde{R}_t$ of $\Gamma(\pi_1 M^3, B)$ with $\|\tilde{R}_t\|_B \le c_1$ where c_1 is a constant independent of t. If we change $l_t = l_t(\gamma_1, \ldots, \gamma_m) \subset \Gamma(\pi_1 M^3, A)$ into the path $\tilde{l}_t \underset{\text{def}}{=} l_t(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$ of $\Gamma(\pi_1 M^3, B)$ we get a B-homotopy connecting λ_n' to λ_n'' and satisfying the following estimates

$$\|\tilde{l}_t\|_B \le (\alpha a_1 n + \eta) b_1. \tag{5.6}$$

$$\|\text{the closed path in } \Gamma(\pi_1 M^3, B) \text{ by which } \tilde{l}_{t+1} \text{ differs from } \tilde{l}_t\|_B \le c_1. \tag{5.7}$$

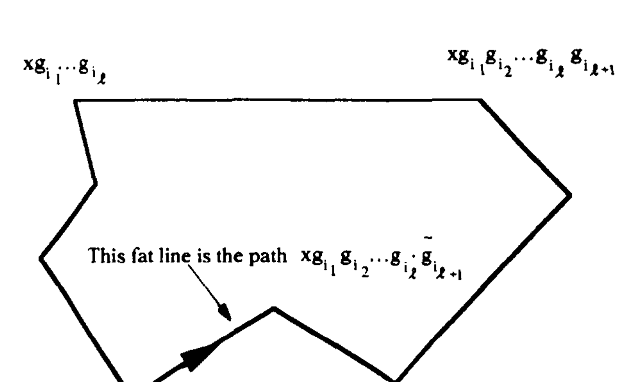

Fig. 2. We see here a closed path of $\Gamma(\pi_1 M^3, B)$ of B-length not exceeding $a_1 b_1 + 1$.

From here on it is a straightforward matter to finish the proof of our Lemma 5.1.

We are now in a position to fix lower bounds for the parameters C_1, C_2, M appearing in the definition of $Z_4(C_1, C_2, M)$. We will fix (a lower bound for) M by asking that $M \geq \tilde{M}$ (see Lemma 5.1) and also that for any of the elements in $\{g_{q+1}^{\pm 1}, g_{q+2}^{\pm 1}, \ldots, g_p^{\pm 1}\} = \mathfrak{B} - B$ we have

$$\|g_{q+i}\|_B \leq M \quad (i = 1, \ldots, p-q). \tag{5.7.1}$$

Lemma 5.1 gives us two quantities $\tilde{C}_1, \tilde{C}_2$ and our theorem also gives us two quantities C, C' (see (1.5)). We will take as values for C_1, C_2 (or at least as lower bounds for such values) the numbers defined by the right-hand side of the following formula (where $n \in Z^+$ is arbitrary)

$$\tilde{C}_1(Cn^{1-\varepsilon} + C' + 4) + \tilde{C}_2 = C_1 n^{1-\varepsilon} + C_2; \tag{5.8}$$

the reason for (5.8) will become apparent quite soon.

Anyway with fixed C_1, C_2, M, chosen large enough so as to fulfil all these requirements, we have our $Z \underset{\text{def}}{=} Z_4(C_1, C_2, M)$ and $G \underset{\text{def}}{=} G(C_1, C_2, M)$ for which we will show that point II of our STRATEGY, namely $\Psi(G) = \Phi(G)$ is also fulfilled (along with I and III).

I will remind the reader at this point that for each RED fundamental domain $\bar{x}\Delta \subset Z_2$, we have a LONG TAIL $\sum_\lambda \bar{x}\lambda$ with which $\bar{x}\Delta$ contributes to Z_3. We will denote by $\bar{y}\Delta$ the generic fundamental domain (possibly a red $\bar{x}\Delta$) of the union of all the long tails $\sum_{\bar{x}\Delta, \lambda} \bar{x}\lambda$. This $\bar{y}\Delta$ contributes to $Z = Z_4$ with its attached SHORT TAIL $\bar{y}T_{\tilde{M}}$ and the initial piece $\bar{y}T_M \subset \bar{y}T_{\tilde{M}}$ will be called the VERY SHORT TAIL. We will use all this in the next lemma.

THE MAIN TECHNICAL LEMMA 5.2 (The closing property for the very SHORT TAILS).

(1) *As an immediate consequence of Lemma* 3.2, *for any fundamental domain* $\bar{y}\Delta \subset \{$*the union of all the* LONG TAILS *corresponding to all the* RED *fundamental domains* $\bar{x}\Delta\}$, *we have* $\Psi(G) \mid \bar{y}T_M = \Phi_M$ *and the map*

$$\bar{y}T_M/\Psi(G) = \bar{y}(T_M/\Phi_M) \xrightarrow{G} y(F(T_M)) \subset \tilde{M}^3,$$

is an isomorphism (*here* $\bar{y}T_M \subset \bar{y}T_{\tilde{M}} \subset Z$).

(2) *Let* $\bar{x}_1\Delta$, $\bar{x}_2\Delta$ *be two red fundamental domains of* Z *such that, in* $\pi_1 M^3$, *we have* $x_1 = x_2 = x$. *Then, the equivalence relation* $\Psi(G)$ *identifies* $\bar{x}_1\Delta$ *to* $\bar{x}_2\Delta$.

(3) *Let* $\bar{x}_1\Delta$, $\bar{x}_2\Delta$ *be two red fundamental domains of* Z *such that, in* $\pi_1 M^3$, *we have* $x_2 = x_1 g_i^{\pm 1}$ *with* $i \leq q$ (*i.e.* $g_i^{\pm 1} \in B$). *Then, the equivalence relation* $\Psi(G)$ *identifies the* $\bar{g}_i^{\pm 1}$*-face of* $\bar{x}_1\Delta$ *to the* $j\bar{g}_i^{\pm 1}$*-face of* $\bar{x}_2\Delta$. *Here* $\bar{g}_i^{\pm 1} \in S$ *is the lift* (3.0) *of* $g_i^{\pm 1}$.

(3*bis*) (*Generalization of* 3) *from* B *to* $\mathfrak{B}$) *Let* $\bar{x}_1\Delta$, $\bar{x}_2\Delta$ *be two red fundamental domains of* Z *such that, in* $\pi_1 M^3$, *we have* $x_2 = x_1 g_k^{\pm 1}$ *with* $k \leq p$. *The equivalence relation* $\Psi(G)$ *identifies the* $g_k^{\pm 1}$*-face of* $\bar{x}_1\Delta$ *to the* $j\bar{g}_k^{\pm 1}$*-face of* $\bar{x}_2\Delta$.

(4) *As a consequence of* (2) *and* (3*bis*), *the subset*

$$\mathscr{R} \underset{\text{def}}{=} \{\text{the union of the red fundamental domains of } Z\}/\Psi(G) \subset Z/\Psi(G)$$

is isomorphic to $\tilde{M}^3$, *via the map* G.

This statement looks superficially like Lemma 4.3 from [16], but the Z^∞ from that lemma is **not** the same object as our present Z. So the proofs are essentially different too. But before proving the main technical fact above, we will show how we can deduct from it that $\Psi(G) = \Phi(G)$, i.e. the requirement II of our STRATEGY. In the context of formula (4.8), we consider the obvious commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, "G"'] & Z/\Psi(G) \arrow[d, "G^1 = \text{IMMERSION}"] \\ & \tilde{M}^3 \end{tikzcd} \tag{5.9}$$

LEMMA 5.3. *For Z, we have $\Psi(G) = \Phi(G)$, and hence G^1 (see 5.9) is an isomorphism between $Z/\Psi(G)$ and $\tilde{M}^3$.*

Proof. We consider the commutative diagram

$$\begin{tikzcd} \mathfrak{R} \arrow[r, "i"] \arrow[dr, "\approx"'] & Z/\Psi(G) \arrow[d, "G^1 = \text{IMMERSION}"] \\ & \tilde{M}^3 \end{tikzcd}$$

where i is the obvious inclusion map. I claim that i is *surjective*. If not, we could find a fundamental domain $\Delta \subset Z/\Psi(G)$ such that $(\operatorname{int}\Delta) \cap \operatorname{Im} i = \phi$. But Z is connected and hence so is $Z/\Psi(G)$. This means that we could also find a Δ with $(\operatorname{int}\Delta) \cap \operatorname{Im} i = \phi \neq (\partial\Delta) \cap \operatorname{Im} i$. But since $\mathscr{R} \xrightarrow{\approx} \tilde{M}^3$ is a homeomorphism, any $x \in (\partial\Delta) \cap \operatorname{Im} i$ would be a singularity for G^1, which is absurd. The conclusion of our lemma follows easily from the surjectivity of the map i.

Proof of the main technical Lemma 5.2. We will consider the "Statement $2^\circ(n)$" obtained by restriction to the x_1, x_2 such that $\|x_1\|, \|x_2\| \leq n$, and similarly the "Statement $3^\circ(n)$" for 3°. We will prove now the implication

$$\{\text{Statements } 2^\circ(n-1) \text{ and } 3^\circ(n-1)\} \Rightarrow \{\text{Statement } 2^\circ(n)\},$$

but, to begin with, we will review some notations from [16].

Let $x \in \pi_1 M^3$ be such that $\|x\| = n$, and like in (4.1), consider $\alpha_1(x)$, $\alpha_2(x)$ (see (4.1)) such that our $\bar{x}_1\Delta$, $\bar{x}_2\Delta$ are the endpoints of the corresponding $\bar{\alpha}_1(x)\Delta$, $\bar{\alpha}_2(x)\Delta$. In Z we have a continuous path of fundamental domains $\bar{\alpha}_1(x)\Delta \cup_{1\Delta} \bar{\alpha}_2(x)\Delta$ with endpoints $\bar{x}_1\Delta$, $\bar{x}_2\Delta$, and what we want to show is that $\Psi(G)$ forces this path to close. We will consider the elements $x(\varepsilon) \in \pi_1 M^3$ (with $\varepsilon = 1, 2$) which are the last ones in $\alpha_\varepsilon(x)$ before x; so $\|x(\varepsilon)\| = n - 1$. For $x(\varepsilon)$ there is in $\Gamma = \Gamma(\pi_1 M^3, B)$ a geodesic path $\alpha(x(\varepsilon))$ *isomorphic to* $\alpha_\varepsilon(x) \mid (n-1)$ and obviously $\Psi(G)$ forces the identification of $\bar{\alpha}(x(\varepsilon))\Delta$ to $(\bar{\alpha}_\varepsilon(x)\Delta) \mid (n-1)$. Now $d_B(x(1), x(2)) \leq 2$ so point 2° from our Theorem tells us that inside the ball of radius $(n-1)$, i.e. in $B(n-1) \subset \Gamma(\pi_1 M^3, B)$ we can join $x(1)$ to $x(2)$ by a path L of length $\leq C(n-1)^{1-\varepsilon} + C'$ (all length, norms, and so on, ... are computed from now in the B-basis, and Γ will stand for Cayley graph $\Gamma(\pi_1 M^3, B)$). In Fig. 3, the path L appears as the fat polygonal line

$$L = (y_0 = x(1), y_1, y_2, \ldots, y_{\bar{m}-1}, y_{\bar{m}} = x(2))$$

To each vertex $v \in \pi_1 M^3$ of λ (with the two end-points x counting as *distinct* vertices), we will attach a *red fundamental domain* in $Z/\Psi(G)$, which we will denote by $\bar{v} \cdot \Delta$ in the following manner.

(i) For the endpoints, we take the images of $\bar{x}_1\Delta, \bar{x}_2\Delta \subset Z$ in $Z/\Psi(G)$.

(ii) As far as the other points $y_i \in B(n-1)$ are concerned, we will use the fact that the inductive hypothesis $2^\circ(n)$ implies that for *any* $w \in B(n-1) \subset \pi_1 M^3$, there is a *unique red* representative in $Z/\Psi(G)$, which we will denote by $\bar{w}\Delta \subset Z/\Psi(G)$: Here $\bar{w}$ is *not* unambiguously defined as an element in the monoid $\bar{\mathscr{G}}$; it is only the *red* fundamental domain $\bar{w}\Delta$

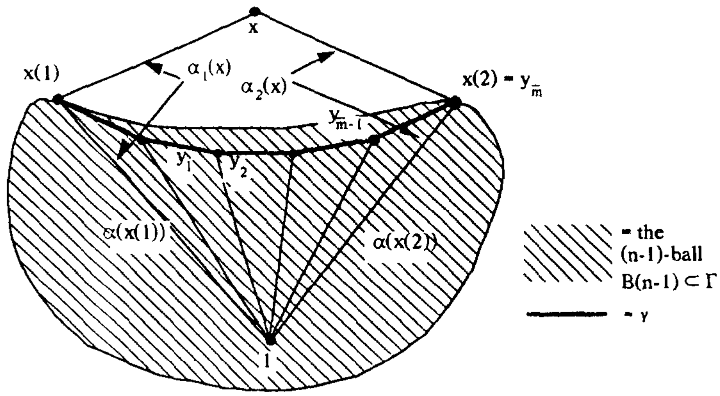

Fig. 3. This figure represents, symbolically, a piece of the Cayley graph Γ. The fat points represent elements of $\pi_1 M^3$. The number $\bar{m}$ is $\leq C(n-1)^{1-\varepsilon} + C'$.

which is well-defined in $Z/\Psi(G)$. If v_{l-1}, v_l are two consecutive vertices of λ, then $[v_{l-1}, v_l]$ corresponds to a well-defined element in $S \subset \mathscr{G}$, which I will denote by $h[v_{l-1}, v_l]$. With this, I *claim* that in $Z/\Psi(G)$, the $h[v_{l-1}, v_l]$-face of $\bar{v}_{l-1}\Delta$ is identified to the $j(h[v_{l-1}, v_l])$-face of $\bar{v}_l\Delta$. For the extremal edges $[x, x(1)]$ and $[x(2), x]$, our claim follows directly from the way in which $\alpha_\varepsilon(x)$, $\alpha(x(\varepsilon))$ were constructed and for all the other edges, the claim follows from the inductive hypothesis $3°(n-1)$. So we get a continuous path Λ of fundamental domains in $Z/\Psi(G)$, joining $\bar{x}_1\Delta$ to $\bar{x}_2\Delta$. We have length$(\Lambda) \leq C(n-1)^{1-\varepsilon} + C' + 2$, and I claim that the equivalence relation $\Psi(G) \subset Z \times Z$ FORCES Λ TO CLOSE.

In order to prove this claim, we start by considering a lift of λ to a path Φ in Z beginning at $\bar{x}_1\Delta$ and continuing in the long tail of $\bar{x}_1\Delta$. The paths Λ and Φ begin at the same fundamental domain and project down to the same path $\lambda \subset \Gamma$. It follows that Λ and Φ are identified by the equivalence relation $\Psi(G)$ and in order to show that Λ closes, it suffices to show that $\Psi(G)$ forces Φ to close.

Notice that $\|\lambda\|_B$ is equal to length $(\Lambda) =$ length(Φ). This means that, at the level of Γ, there is a B-homotopy λ_t connecting λ to the trivial path at x and which is such that

$$\|\lambda_t\|_B \leq \tilde{C}_1(C(n-1)^{1-\varepsilon} + C' + 2) + \tilde{C}_2 \leq C_1 n^{1-\varepsilon} + C_2.$$

One uses here our estimate for length (Λ) and Lemma 5.1; the inequality (5.4) will also hold for our B-homotopy.

We can lift λ_t to a sequence of paths of fundamental domains Φ_t all beginning at $\bar{x}_1\Delta$, continuing in the long tail of $\bar{x}_1\Delta$, and connecting the path Φ to the trivial path reduced to $\bar{x}_1\Delta$. The paths λ_{i-1}, λ_i are factorized like in (5.2) and these factorizations satisfy the estimate (5.4).

Corresponding to the factorizations (5.2) we will have

$$\Phi_{i-1} = \bar{\alpha}_-(i)\cdot\bar{\beta}_{i-1}(i)\cdot\bar{\gamma}_-(i), \quad \Phi_i = \bar{\alpha}_+(i)\cdot\bar{\beta}_i(i)\cdot\bar{\gamma}_+(i).$$

Since $\bar{\alpha}_-(i)$, $\bar{\alpha}_+(i)$ both start at $\bar{x}_1\Delta$ and project down to the same $\alpha_i \subset \Gamma$, the equivalence relation $\Psi(G)$ has to identify them. The short tail at the endpoint of $\bar{\alpha}_-(i)$ corresponding to x_i, forces then the identification of the endpoints of $\bar{\beta}_i(i)$, $\bar{\beta}_{i-1}(i)$ which correspond to y_i; one uses here the closing property for the very short tails, the estimate (5.4) and the fact that $M \geq \tilde{M}$.

It is easy to show now that $\Psi(G)$ has to identify the endpoints of $\bar{\gamma}_-(i)$, $\bar{\gamma}_+(i)$ hence the endpoints of Φ_{i-1}, Φ_i and, inductively, that $\Psi(G)$ forces Φ to close.

All this proves the implication

$$\{\text{Statements } 2^\circ(n-1) \text{ and } 3^\circ(n-1)\} \Rightarrow \{\text{Statement } 2^\circ(n)\}$$

and the same line of argument can be used to prove that

$$\{\text{Statements } 2^\circ(n-1) \text{ and } 3^\circ(n-1)\} \Rightarrow \{\text{Statement } 3^\circ(n)\}.$$

(The analog of Λ will have now a length $\leq C(n-1)^{1-\varepsilon} + C' + 3$ and so the full strength of (5.8) will be needed).

We leave it to the reader to complete the proof of (3) following this line of argument. In order to show that (2) and (3) together imply (3bis), we consider the red fundamental domains $\bar{x}_1\Delta, \bar{x}_2\Delta \subset Z$ with $x_2 = x_1 g_k$ $(q < k \leq p)$ in $\pi_1 M^3$. At the level of $\tilde{M}^3$, the fundamental domains $G^1(\bar{x}_1\Delta), G^1(\bar{x}_2\Delta)$ touch along their respective (g_k and g_k^{-1})-faces and we want to show that $\Psi(G)$ forces them actually to be *glued* together at that site, at the level of $Z/\Psi(G)$, source of the map G^1.

In $\pi_1 M^3$, we can express g_k as a word with letters from B, of length $\leq M$. This gives rise, at the level of $Z/\Psi(G)$, to a continuous path of fundamental domains Λ', which connects $\bar{x}_1\Delta$ to $\bar{x}_2\Delta$.

Now, in Z, the *red* fundamental domains $\bar{x}_1\Delta, \bar{x}_2\Delta$ come from Z_2, but Z *also* contains a piece $\bar{x}_1 T_{\bar{M}} \supset \bar{x}_1 T_M$ (i.e. the SHORT TAIL and the VERY SHORT TAIL of $\bar{x}_1\Delta$) whose own $\bar{x}_1\Delta$ is glued (at the level of Z) to our original *red* $\bar{x}_1\Delta$. If we consider the obvious commutative diagram

$$\begin{tikzcd} \bar{x}_1 T_{M_1} \cup_{\bar{x}_1 \cdot \Delta} \Lambda' \arrow[rr, "\zeta"] \arrow[dr] & & Z/\Psi(G) \arrow[dl, "G^1"] \\ & \tilde{M}^3 & \end{tikzcd}$$

we can make the following remarks. Our Λ' which is of length $\leq M$ corresponds canonically to a piece of $\bar{x}_1 T_M \subset \bar{x}_1 T_{\bar{M}}$ and at the level of $Z/\Psi(G)$ the two corresponding ζ-images *have to be identified* to each other. Point (1) of our lemma tells us, on the other hand, that $G^1 \mid \zeta(\bar{x}_1 T_M)$ is *injective*. This means that in $Z/\Psi(G)$ the image $\zeta(\Lambda')$ is a *closed path.*

This proves point (3 bis) of our MAIN TECHNICAL LEMMA, and point (4) is left to reader.

Acknowledgements—This paper owes a lot to conversations and/or letter exchanged with S. Gersten, H. Short and J. Stallings. I also want to thank the referee for his suggestion of how to simplify the proof of Lemma 5.2, and C. Tanasi for his friendly help with the typing.

REFERENCES

1. M. Bestvina and G. Mess: The boundary of negatively curved groups, Preprint (1990).
2. J. W. Cannon: Almost convex groups, *Geometria Dedicata* **22** (1987), 197–210.
3. M. Coornaert, T. Delzant and A. Papadopoulos: *Notes sur les groupes hyperboliques de Gromov (chapitres 1 à 12)*, Publications IRMA (1990).
4. M. Domergue: Extension du lemme de Dehn, Note *CRAS* **28** (1978), 885–887.
5. L. Funar: Discrete cocompact solvgroups have strange balls, *Prépublications d'Orsay*, 91–15 (1991).
6. S. M. Gersten: The linear isodiametric inequality for groups and 3-manifolds, Preprint (1990).
7. S. M. Gersten: Lattices in SOL satisfy an isodiametric inequality, Preprint (1990).
8. E. Ghys, P. de la Harpe (Editors): Sur les groupes hyperboliques d'après Mikhael Gromov, PM 83 Birkhäuser (1990).
9. M. Gromov: Infinite groups as geometric objects, *Proceedings ICM*, Warszewa (1983), 385–392.
10. M. Gromov: *Hyperbolic groups, in Essays in group theory*, S. M. Gersten Ed., MSRI Publications 8 (1987), 75–263.
11. C. D. Papakyriakopoulos: On Dehn's lemma and the asphericity of knots, *Ann. Math.* **1** (1957), 1–26.

12. V. Poénaru and C. Tanasi: Hausdorff combing of groups and π_1^∞ for universal covering spaces of closed 3-manifolds, *Prépublications d'Orsay* 90-43 (1990).
13. V. Poénaru: The equivalence relation forced by the singularities of a nondegenerate simplicial map, *Duke Math. J.*, **63** (1991), 421–430.
14. V. Poénaru: Killing handles of index one stably and π_1^∞, *Duke Math. J.*, **63** (1991), 431–437.
15. V. Poénaru: The collapsible pseudo-spine representation theorem, *Topology*, **31** (1992), 625–656.
16. V. Poénaru: Almost convex groups, Lipzchitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds, *J. Diff. Geom.* **35** (1992), 103–130.
17. V. Poénaru: A general finiteness theorem in geometric group theory, *Prépublications d'Orsay* 92-33 (1992).
18. V. Poénaru: Representations of open simply connected 3-manifolds, a finiteness result, *Prépublications d'Orsay* 92-69 (1992).
19. A. Shapiro and J. H. C. Whitehead: A proof of extension of Dehn's Lemma, *BAMS* **64** (1958), 174–178.
20. M. Shapiro: A geometric approach to the almost convexity and growth of some nilpotent groups, *Math. Ann.* **285** (1989), 601–624.
21. M. Shapiro: Preprint.
22. H. Short: Groupes peignables, d'après Poénaru, Casson et Stallings, Preprint ENS Lyon (1990).
23. J. Stallings and M. Gersten: Casson's idea about 3-manifolds whose universal cover is R^3, Preprint (1991).
24. D. Sullivan and W. Thurston: Manifolds with canonical coordinate charts, some examples, *l'Enseign. Math.* **29** (1983), 15–25.
25. W. P. Thurston: The geometry and topology of 3-manifolds, to appear.

Université de Paris-Sud
Mathématique Bâtiment 425
91405 *Orsay cedex*
France